HIGH SCHOOL MATHEMATICS CONTESTS

Math League Press, P.O. Box 17, Tenafly, New Jersey 07670-0017

All official participants must take this contest at the same time.

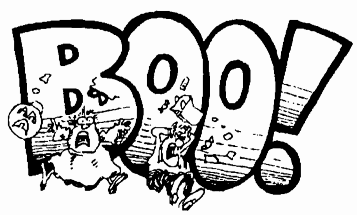
Contest Number 1 *Any calculator without a QWERTY keyboard is allowed.* Answers must be exact *or* have 4 (or more) significant digits, correctly rounded. **October 25, 2005**

Name ______________________ **Teacher** ___________ **Grade Level** _____ **Score** ____

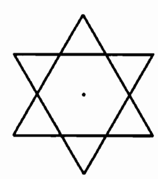
Time Limit: 30 minutes NEXT CONTEST: NOV. 29, 2005 *Answer Column*

1-1. If $a < b$, then $3^2+4^2+5^2+12^2 = a^2+b^2$ is satisfied by only one pair of positive integers (a,b). What is the value of $a+b$? — 1-1.

1-2. This coming Halloween, Tom plans to scare twice as many people as Sam, and Sam plans to scare three times as many people as Roz. In all, they plan to scare at most 2005 people. If no one is scared more than once, at most how many people does Sam plan to scare? — 1-2.

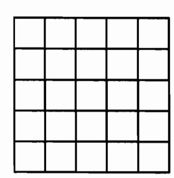


1-3. When two congruent equilateral triangles share a common center, their union can be a star, as shown. If their overlap is a regular hexagon with an area of 60, what is the area of one of the original equilateral triangles? — 1-3.



1-4. For how many different positive integers n does $\sqrt{n}$ differ from $\sqrt{100}$ by less than 1? — 1-4.

1-5. Counting every possible square of each size from 1×1 to 5×5, what is the total number of distinct squares which can be traced out along the lines of the accompanying grid? — 1-5.



1-6. The four numbers $a < b < c < d$ can be paired in six different ways. If each pair has a different sum, and if the four smallest sums are 1, 2, 3, and 4, what are all possible values of d? — 1-6.

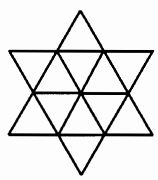
Problem 1-1

$3^2+4^2 \;+\; 5^2+12^2 = 5^2+13^2$, so $a+b = \boxed{18}$.

Problem 1-2

If Roz plans to scare n people, then Sam plans to scare $3n$ people and Tom plans to scare double that, $6n$. Altogether, $n+3n+6n \le 2005$; so $n \le 200.5$. Since n is an integer, n is at most 200. Since Sam plans to scare $3n$ people, that's at most $\boxed{600}$ people.

Problem 1-3



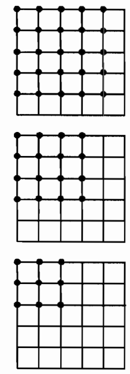
Method I: Draw the 3 diagonals of the hexagon, as shown, to partition the figure into 12 congruent small equilateral triangles. Since the overlap consists of 6 of these triangles, with a total area of 60, each small equilateral triangle has an area of 10. Since each original (large) equilateral triangle consists of 9 small ones, the area of one large equilateral triangle is $\boxed{90}$.

Method II: Join the common center to any two consecutive vertices of the hexagon. The equilateral triangle inside the hexagon is congruent to each equilateral triangle outside the hexagon. Since six of these "inside" triangles make up the hexagon, a large equilateral triangle consists of these six triangles plus three additional triangles. Therefore, the area of a large equilateral triangle is $60 + (1/2)(60) = 90$.

Problem 1-4

If the integer n is greater than $9^2 = 81$ but less than $11^2 = 121$, then $\sqrt{n}$ differs from 10 by less than 1. The set $\{82, 83, \ldots, 119, 120\}$ contains $\boxed{39}$ different integers.

Problem 1-5



Each 1×1 square is determined by its upper left vertex, for which there are 5×5 choices. Each 2×2 square is determined by its upper left vertex, for which there are 4×4 choices. Each 3×3 square is determined by its upper left vertex, for which there are 3×3 choices. In general, each $d\times d$ square is determined by its upper left vertex, for which there are $(6-d)(6-d) = (6-d)^2$ choices. Thus, the number of 1×1 squares is 5^2, the number of 2×2 squares is $4^2, \ldots$, and the number of 5×5 squares is 1^2. The total of the number of squares of all sizes is $5^2+4^2+3^2+2^2+1^2 = \boxed{55}$.

Problem 1-6

Method I: The six possible sums are $a+b$, $a+c$, $b+c$, $a+d$, $b+d$, and $c+d$. Since $a < b < c < d$, the smallest sums are $a+b = 1$ and $a+c = 2$. From these, $c = b+1$. Of the other four sums, the largest are $b+d$ and $c+d$. Of the remaining sums, $b+c$ and $a+d$, one equals 3 and the other equals 4. Since $c = b+1$, if $b+c = 2b+1 = 3$, then $b = 1$. Then, since $a+b = 1$, $a = 0$; and since $a+d = 4$, $d = 4$. If, instead, $b+c = 2b+1 = 4$, then $b = 3/2$. Then, since $a+b = 1$, $a = -1/2$; and since $a+d = 3$, $d = 7/2$. Finally, the two possible values of d are $\boxed{7/2,\ 4}$.

Method II: The six possible sums are $a+b$, $a+c$, $b+c$, $a+d$, $b+d$, and $c+d$. Since $a < b < c < d$, the smallest two sums are $a+b = 1$ and $a+c = 2$. From these, $b = 1-a$ and $c = 2-a$. Of the other four sums, the largest are $b+d$ and $c+d$. Of the remaining sums, $b+c$ and $a+d$, one equals 3, the other equals 4. If $b+c = 3$, then $(1-a)+(2-a) = 3$. Solving, $a = 0$. Since $a+d = 4$, $d = 4$. If $b+c = 4$ and $a+d = 3$, then $(1-a)+(2-a) = 4$, so $a = -1/2$ and $d = 7/2$.

Contests written and compiled by Steven R. Conrad & Daniel Flegler